4. Optimum motion control of redundant serial manipulators

4.1 How to calculate Degrees-of-Freedom (DOF)

Grubler’s equation:

\[ F = F_S (N - 1) - \sum_{f=1}^{F_S - 1} (F_S - f) J_f \]

- \( F \): DOF of mechanism
- \( F_S \): DOF of space
- \( N \): Number of links
- \( f \): DOF of pairs
- \( J_f \): Number of pairs with \( f \) DOF
**Ex. Planar 2R manipulator**

Non redundant robot!

Grubler’s equation:

\[
F = F_s (N - 1) - \sum_{f=1}^{s-1} (F_s - f)J_f
\]

2 outputs to be control

\[
= 3(3-1) - 3(3-1) \cdot 2
\]

= 2 DOF mechanism

\[
F_s = 3 \quad \text{Planar motion}
\]

\[
N = 3 \quad \text{Including a frame}
\]

\[
f = 1 \quad \text{Revolute pair (1 DOF)}
\]

\[
J_f = 2 \quad 2 \text{ Revolute pairs}
\]

2 active pairs (Actuators)

**Ex. Planar 3R manipulator**

Redundant robot!

(1 redundant DOF)

Grubler’s equation:

\[
F = F_s (N - 1) - \sum_{f=1}^{s-1} (F_s - f)J_f
\]

2 outputs to be control

\[
= 3(4-1) - 3(4-1) \cdot 3
\]

= 3 DOF mechanism

\[
F_s = 3 \quad \text{Planar motion}
\]

\[
N = 4 \quad \text{Including a frame}
\]

\[
f = 1 \quad \text{Revolute pair (1 DOF)}
\]

\[
J_f = 3 \quad 3 \text{ Revolute pairs}
\]

3 active pairs (Actuators)
### 4.2 Forward kinematics

**Ex. Planar 2R manipulator (Non-redundant)**

**Output displacement** $r$

$$R(\text{X, Y})$$

**Angular displacement**

$$r(\theta_1, \theta_2) = \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X_1 + L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) \\ Y_1 + L_1 \sin \theta_1 + L_2 \sin(\theta_1 + \theta_2) \end{bmatrix}$$

**Output velocity** $\dot{r}$

$$\dot{r}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = \begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix}$$

$$= \begin{bmatrix} -L_1 \dot{\theta}_1 \sin \theta_1 - L_2 (\dot{\theta}_1 + \dot{\theta}_2) \sin(\theta_1 + \theta_2) \\ L_1 \dot{\theta}_1 \sin \theta_1 + L_2 (\dot{\theta}_1 + \dot{\theta}_2) \cos(\theta_1 + \theta_2) \end{bmatrix}$$

$$= \begin{bmatrix} -L_1 \sin \theta_1 - L_2 \sin(\theta_1 + \theta_2) - L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$= J \dot{\theta}$$

**Jacobian matrix**

(2x2 Matrix)
**Ex. Planar 3R manipulator (Redundant)**

**Output displacement** \( \mathbf{r} \)

\[
\mathbf{r}(\theta_1, \theta_2, \theta_3) = \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X_1 + L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3) \\ Y_1 + L_1 \sin \theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3) \end{bmatrix}
\]

**Displacement**

**Velocity**

\[
r(\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3) = \begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = \begin{bmatrix} -L_1 \dot{\theta}_1 \sin \theta_1 - L_2 \dot{\theta}_2 \sin(\theta_1 + \theta_2) - L_3 \dot{\theta}_3 \sin(\theta_1 + \theta_2 + \theta_3) \\ L_1 \dot{\theta}_1 \sin \theta_1 + L_2 \dot{\theta}_2 \cos(\theta_1 + \theta_3) + L_3 \dot{\theta}_3 \cos(\theta_1 + \theta_2 + \theta_3) \end{bmatrix}
\]

\[
= \begin{bmatrix} -L_1 \sin \theta_1 - L_2 \sin(\theta_1 + \theta_2) - L_3 \sin(\theta_1 + \theta_2 + \theta_3) \\ L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}
\]

= \( \mathbf{J} \mathbf{\dot{\theta}} \)

**Jacobian matrix**

(2x3 Matrix)
4.3 Inverse kinematics

Ex. Planar 2R manipulator (Non-redundant)

Desired output displacement $r$

![Diagram of a planar 2R manipulator with input angular displacement and input joint angles becoming settled uniquely.]

\[ \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \alpha - \beta \\ \pi - \gamma \end{bmatrix} \]

\[ L = \sqrt{(X - X_1)^2 + (Y - Y_1)^2} \]

Input joint angles become settled uniquely.

Input angular displacement

Input joint angular velocities become settled uniquely.

Inverse of Jacobian matrix (2x2 Matrix)

Desired output velocity $\dot{r}$

\[ \dot{\theta} = J^{-1} \dot{r} \]
Ex. Planar 3R manipulator (Redundant)

Output displacement \( r \)
\( R(X, Y) \)

Input angular displacement

Since this robot has redundant 1 DOF, one joint input should be specified!

Posture angle of end-effector, \( \Phi \), is assumed, the joint position \( J_3 \) is then calculated as

\[
J_3 = \begin{bmatrix} X_3 \\ Y_3 \end{bmatrix} = \begin{bmatrix} X - L_3 \cos \phi \\ Y - L_3 \sin \phi \end{bmatrix}
\]

Through same procedure as 2R-manipulator

Input joint angles are determined as a function of \( \phi \).
Ex. Planar 3R manipulator (Redundant)

Desired output velocity \( \dot{r} \)

\[ R(X, Y) \]

\( \theta_1 \)

\( L_1 \)

\( L_2 \)

\( L_3 \)

\( \theta_2 \)

Input joint angular velocities are determined as a function of vector \( \alpha \).

\[ \dot{\theta} = J^\# \dot{r} + (I - J^\# J) \alpha \]

\[ J^\# = (JJ^T)^{-1} \]

\( \alpha \)

Input angular velocity

\( J \dot{\theta} = J[J^\# \dot{r} + (I - J^\# J)\alpha] \)

\[ = J[J^T (JJ^T)^{-1} \dot{r} + [I - J^T (JJ^T)^{-1}] J] \alpha \] \( = \dot{r} \)

4.4 Optimum motion control

In order to utilize the redundant DOF, a certain objective function should be minimized or maximized by adjusting redundant inputs \( \phi \) or \( \alpha \) while the robot generates the desired output motion \( r \) or \( \dot{r} \).

Various objective functions:

(1) Kinematical performances
   - Maximize speed
   - Dexterous motion

(2) Statics/dynamics performances
   - Maximize output force
   - Minimize actuator torques
4.5 Objective function on dexterity

How to evaluate dexterity

Dexterity = Ability to achieve complicated motion

1) Evaluation based on workspace

Limitation of end-effector motion

- Workspace
- Accessible workspace
  ○ Easy to understand
    (How can end-effector reach desired position?)
  ▲ Difficult to calculate
  ▲ Strongly dependent on end-effector length

2) Evaluation based on manipulability

Feasible velocity of end-effector

- Manipulability measure
  - Condition number of Jacobian matrix
    ○ Easy to calculate
    ▲ Difficult to understand

4.6 Manipulability measure

Ellipsoid drawn by the output velocities
(Input joint velocity will be linearly converted with $J_p$)

Joint angular velocity $\dot{\theta}$
Serial manipulator

Manipulability measure:
(Proposed by T. Yoshikawa)

$$w_p = \begin{cases} 
\det J_p & \text{(Non-redundant)} \\
\sqrt{\det (J_p \cdot J_p^T)} & \text{(Redundant)}
\end{cases}$$

$J_p$: Jacobian matrix
  ($\dot{r} = J_p \dot{\theta}$)

"Proportional to feasible velocities in all directions"
**Ex. Planar 2R manipulator (Non-redundant)**

\[ \mathbf{r}_p = \begin{bmatrix} X \\ Y \\ \phi \end{bmatrix} = \begin{bmatrix} L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) \\ L_1 \sin \theta_1 + L_2 \sin(\theta_1 + \theta_2) \\ \theta_1 + \theta_2 \end{bmatrix} \]

\[ \dot{\mathbf{r}}_p = \mathbf{J}_p \dot{\mathbf{\theta}} \]

\[ \mathbf{J}_p = \begin{bmatrix} -L_1 \sin \theta_1 - L_2 \sin(\theta_1 + \theta_2) & -L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) & L_2 \cos(\theta_1 + \theta_2) \\ \theta_1 + \theta_2 & 1 \end{bmatrix} \]

By taking account of end-effector’s posture, this robot is assumed underactuator mechanism.

\[ w_p = 0 \]

*The 2R-serial manipulator has zero dexterity.*

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**Ex. Planar 3R manipulator (redundant)**

\[ \mathbf{r}_p = \begin{bmatrix} X \\ Y \\ \phi \end{bmatrix} = \begin{bmatrix} L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3) \\ L_1 \sin \theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3) \\ \theta_1 + \theta_2 + \theta_3 \end{bmatrix} \]

\[ \dot{\mathbf{r}}_p = \mathbf{J}_p \dot{\mathbf{\theta}} \]

\[ \mathbf{J}_p = \begin{bmatrix} -L_1 \sin \theta_1 - L_3 \sin(\theta_1 + \theta_2) - L_3 \sin(\theta_1 + \theta_2 + \theta_3) & -L_2 \sin(\theta_1 + \theta_2) - L_3 \sin(\theta_1 + \theta_2 + \theta_3) & -L_3 \sin(\theta_1 + \theta_2 + \theta_3) \\ L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3) & L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3) & L_3 \cos(\theta_1 + \theta_2 + \theta_3) \\ \theta_1 + \theta_2 + \theta_3 & 1 & 1 \end{bmatrix} \]

\[ w_p = \frac{\det \mathbf{J}_p}{\det J_p} = L_1 L_2 [\cos \theta_1 \sin(\theta_1 + \theta_2) - \sin \theta_1 \cos(\theta_1 + \theta_2)] = L_1 L_2 \sin \theta_1 \]

Therefore, the manipulator can take \( \theta_2 = \pi/2 \) , \( w_p \) takes it maximum

\[ w_{p,max} = L_1 L_2 \]
4.7 Optimum configuration to maximize dexterity

For 3R-manipulator (1 redundant DOF), the configuration to reach desired output position can be calculated by giving $\phi$.

To search optimum $\phi$ to maximize $w_p$
(One-dimensional optimization)

As same procedure:
For 4R-manipulator (2 redundant DOF), the configuration to reach desired output position can be calculated by giving $\phi_1, \phi_2, \ldots$.

To search optimum $\phi_1, \phi_2$, to maximize $w_p$
(Two-dimensional optimization)

\ldots

For nR-manipulator

Nonlinear simplex method is used for optimization.

Tested manipulators with quite same workspace

2R : $\ell_1 : \ell_2 = 1 : 1$
3R : $\ell_1 : \ell_2 : \ell_3 = 1 : 0.5 : 0.5$
4R : $\ell_1 : \ell_2 : \ell_3 : \ell_4 = 1 : 0.5 : 0.25 : 0.25$
5R : $\ell_1 : \ell_2 : \ell_3 : \ell_4 : \ell_5 = 1 : 0.5 : 0.25 : 0.125 : 0.125$
The maximum manipulability measure of serial planar manipulators

Dexterity increases as redundancy.

4.8 Motion control experiments

Experimental prototype (4R-manipulator)
Experimental prototype (4R-manipulator)

Block diagram of servosystem
Dexterous configuration of 3R-manipulator

\[ \theta_2 = \pi/2 \]

Dexterous configuration of 4R-manipulator
Dexterous configuration of 5R-manipulator

Tested trajectories
Result of the optimum CP control (4R, trajectory B)

Discontinuity of velocities

Maximum manipulability

Result of the optimum CP control (4R, trajectory B)
Dexterity per redundant DOF

Time ratio to keep desired trajectory

Operating speed

Performance on response
(4R-manipulator)
Performance on response to various manipulators

Operating speed

Time ratio to keep desired trajectory

Positioning error due to velocity discontinuity

Results of optimum CP control in case where posture angle of end-effector is continuously specified

Input joint angles become smooth.

4.9 Improved optimum motion control

Specified as continuous function
Result of the optimum CP control in case where posture angle of end-effector is continuously specified (4R, trajectory B)

Manipulability decreases a little.

Manipulability measure in the optimum CP control in case where posture angle of end-effector is continuously specified (4R-manipulator)
Performance on response in the optimum CP control in case where posture angle of end-effector is continuously specified (various manipulator)

4.10 Concluding remarks on optimum motion control of serial redundant manipulators

(1) Inverse kinematics can be calculated for displacement/velocity inputs by specifying redundant DOF.

(2) Manipulability measure taking account of posture angle of end-effector is suitable to evaluate dexterous manipulation.

(3) Optimum motion control to maximize the proposed dexterity measure was achieved.

(4) Dexterity increases as redundant DOF.

(5) By specifying smooth posture angle of end-effector, redundant robots achieve quick response.